

ON SHIMURA CURVES IN THE SCHOTTKY LOCUS

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ABSTRACT. We show that a given rational Shimura curve Y with strictly maximal Higgs field in the moduli space of g -dimensional Abelian varieties does not generically intersect the Schottky locus for large g .

We achieve this by using a result of Viehweg and Zuo which says that if Y parameterizes a family of curves of genus g , then the corresponding family of Jacobians is Y -isogenous to the g -fold product of a modular family of elliptic curves. After reducing the situation from the field of complex numbers to a finite field, we will see, combining the Weil and Sato-Tate conjectures, that this is impossible for large g .

CONTENTS

1. Introduction	1
2. Shimura curves and maximal Higgs fields	3
3. Reduction to number fields	5
4. Reduction to finite fields	10
5. Bounding the genus in characteristic p	17
6. Conclusion	21
References	23

1. INTRODUCTION

Let U be a rational Shimura curve so that U is an open subset in $\mathbb{P}_{\mathbb{C}}^1$. Let $S = \mathbb{P}_{\mathbb{C}}^1 - U$ be the “bad locus”. We prove the following theorem.

Theorem 1.1 (Shimura curves in the Schottky locus). *Given an integer $s \geq 0$, there is a natural number $B = B(s)$, depending only on s , such that a rational Shimura curve, whose Higgs field is strictly maximal and whose bad locus S contains at most s points, cannot lie in the closure of the Schottky locus \mathcal{M}_g for $g > B$.*

The Schottky locus is the image of the moduli space \mathcal{M}_g of curves of genus g in \mathcal{A}_g - the moduli space of principally polarized g -dimensional Abelian varieties (with a suitable level structure). We say that a Shimura curve in \mathcal{A}_g lies in the closure of the Schottky locus if it generically intersects the image of \mathcal{M}_g in \mathcal{A}_g . So, in particular, it should not lie entirely in the boundary of the closure of \mathcal{M}_g in \mathcal{A}_g .

By Shimura variety we mean a Shimura variety of Hodge type which is an étale covering of a certain moduli space of Abelian varieties with prescribed Mumford-Tate group and a suitable level structure as defined in [Mu66]. A Shimura curve is a one-dimensional Shimura variety.

Let $f : A \rightarrow Y$ be a semistable family of Abelian varieties over a complex projective curve Y , $U = Y - S$ the smooth locus and $V = f^{-1}(U)$ so that $f : V \rightarrow U$ is an Abelian scheme. Consider the Higgs bundle (E, θ) given by taking the graded sheaf of the Deligne extension of $R^1 f_* \mathbb{C}_V \otimes \mathcal{O}_U$ where $R^1 f_* \mathbb{C}_V$ is the weight 1 variation of Hodge structures. We have a decomposition $E = F \oplus N$ into an ample part F and a flat part N . Following [VZ03] we say that the Higgs field is *maximal* if

$$\theta^{1,0} : F^{1,0} \longrightarrow F^{0,1} \otimes \Omega_Y^1(\log S)$$

is an isomorphism, and that the Higgs field is *strictly maximal* if additionally $N = 0$.

Viehweg and Zuo showed in [VZ04] that if each irreducible and non-unitary sub-variation \mathbb{V} of Hodge structures in $R^1 f_* \mathbb{C}_V$ has a strictly maximal Higgs field, then there is an étale covering $U' \rightarrow U$ such that U' is a Shimura curve and $f' : V' \rightarrow U'$ is the corresponding universal family. Moreover, Möller showed in [Mö05] that the converse also holds. Hence we have a characterization of Shimura curves by the maximality of the Higgs field of the corresponding universal family.

Combining the results of Viehweg and Zuo [VZ06] which say that a Shimura curve U in \mathcal{M}_g has to be non-compact with the techniques of Möller shows that U has also to be a Teichmüller curve. Then from [Mö05] it follows that there are no such curves in \mathcal{M}_g unless $g = 3$. See also the discussion in [MVZ05].

Observe that this result deals with the occurrence of Shimura curves in \mathcal{M}_g rather than its closure in \mathcal{A}_g . So it does not answer the question if there are Shimura curves in the closure of the Schottky locus.

We remark that the conjecture of André-Oort, saying that a Shimura variety is characterized by having a dense set of CM-points, and the conjecture of Coleman, saying that there are only finitely many CM-points in \mathcal{M}_g for high genus g , suggest that there are no Shimura varieties in the closure of the Schottky locus for g sufficiently large.

In [Ha99], Hain studied families of compact Jacobians over locally symmetric domains U satisfying an additional technical condition. Based on his methods, de Jong and Zhang [dJZ06] were able to exclude certain types of higher-dimensional Shimura varieties.

Returning to Shimura curves, we will prove Theorem (1.1) as follows. Let $C \rightarrow Y$ be a family of complex curves whose family of Jacobians $J \rightarrow Y$ has a strictly maximal Higgs field. If $Y = \mathbb{P}_{\mathbb{C}}^1$, then a further result of Viehweg and Zuo from [VZ04] says that $J \rightarrow Y$ is Y -isogenous to the g -fold product $E \times_Y \cdots \times_Y E$ of a modular family of elliptic curves $E \rightarrow Y$. So Theorem

(1.1) will follow from the following theorem, which holds for an arbitrary base curve Y .

Theorem 1.2 (Bound for the genus). *Let $C \rightarrow Y$ be a family of curves of genus g whose Jacobian $J \rightarrow Y$ is Y -isogenous to the g -fold product of a non-isotrivial family of elliptic curves $E \rightarrow Y$ which can be defined over a number field. Then the genus g is bounded, i. e. there is a number $B = B(E/Y)$, depending only on $E \rightarrow Y$, such that g is smaller than B .*

Mind that modular families of elliptic curves can be defined over number fields. We will prove Theorem (1.2) by reducing the situation from \mathbb{C} to a number field F . Then we reduce to a finite field by selecting a suitable finite prime of F . Finally, we prove that the genus g of the fibers of a family of curves $C \rightarrow Y$, defined over a finite field and whose family of Jacobians is isogenous to the g -fold product of a family of elliptic curves, is bounded.

We achieve this by counting the number of singularities δ in the fibers of $C \rightarrow Y$. Combining the Weil conjectures for the fibers with the Sato-Tate conjecture about the distribution of Frobenius traces in a family of elliptic curves, we will get a lower bound for δ . On the other hand, the geometry of the total space C of $C \rightarrow Y$ will give an upper bound. For large g , the lower bound will exceed the upper bound. Thus, the genus has to be bounded.

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2. SHIMURA CURVES AND MAXIMAL HIGGS FIELDS

Let Y be an irreducible smooth projective curve over the field of complex numbers \mathbb{C} and let $f : A \rightarrow Y$ be a semistable family of g -dimensional Abelian varieties, i. e. $f : A \rightarrow Y$ is a flat, projective \mathbb{C} -morphism whose generic fiber is an Abelian variety of dimension g . Let $U \subset Y$ be the smooth locus of $A \rightarrow Y$, i. e. the restriction of $A \rightarrow Y$ to U is an Abelian scheme $A_0 \rightarrow U$ while the fibers over the set $S = Y - U$ are all singular. Consider the weight 1 variation of Hodge structures $R^1 f_* \mathbb{Z}_{A_0}$ and let F be the non-flat part of the Higgs bundle (E, θ) given by taking the graded sheaf of the Deligne extension of $R^1 f_* \mathbb{Z}_{A_0} \otimes \mathcal{O}_U$ to Y which carries a Hodge filtration. Then the Arakelov inequality for families of Abelian varieties [JZ02] says that

$$0 < 2 \cdot \deg(F^{1,0}) \leq g_0 \cdot (2g - 2 + \#S)$$

where g denotes the genus of the base curve Y and g_0 is the rank of $F^{1,0}$. We say that the family of Abelian varieties $A \rightarrow Y$ *reaches the Arakelov bound* if the above inequality becomes an equality. Viehweg and Zuo showed in [VZ04] that this property is equivalent to the *maximality* of the Higgs field for F , i. e. the map $\theta|_{F^{1,0}} : F^{1,0} \rightarrow F^{0,1} \otimes \Omega_Y^1(\log S)$ is an isomorphism.

Moreover, the Higgs field is called *strictly maximal* if in addition the Higgs bundle has no flat part.

Assume that $V \rightarrow U$ is a Shimura curve, i.e. U is an étale covering of a certain moduli space of Abelian varieties with prescribed Mumford-Tate group and a suitable level structure, and $V \rightarrow U$ is the corresponding universal family, see [Mö05]. Then Möller showed that $U \rightarrow V$ has a maximal Higgs field.

Theorem 2.1 (Shimura implies maximal Higgs). *If $V \rightarrow U$ is the universal family over a Shimura curve, then its Higgs field is maximal.*

Proof. See [MVZ05, Thm.0.9] or [Mö05, Thm.1.2]. □

The converse was shown by Viehweg and Zuo in [VZ04], see also [MVZ05]. So we have a Characterization of Shimura curves by its corresponding Higgs field. Moreover, Viehweg and Zuo showed that after an étale extension the family $A \rightarrow Y$ decomposes in the following way.

Theorem 2.2 (Decomposition Theorem). *If $V \rightarrow U$ has a maximal Higgs field and $S \neq \emptyset$, then there is an étale covering $Y' \rightarrow Y$ such that the pull-back family $A' \rightarrow Y'$ is Y' -isogenous to a product*

$$E \times_{Y'} \cdots \times_{Y'} E \times_{\mathbb{C}} B$$

where B/\mathbb{C} is an Abelian variety of dimension $g - g_0$ and $E \rightarrow Y'$ is a modular family of elliptic curves.

Proof. See [VZ04, Thm.0.2] or [MVZ05, Cor.0.10]. □

Modular means that the smooth locus U' of $E \rightarrow Y'$ is the quotient $\Gamma \backslash \mathbb{H}$ of the upper half-plane \mathbb{H} by a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ of finite index and $E \rightarrow Y'$ is over U' the quotient of $\mathbb{H} \times \mathbb{C}$ by the semi-direct product of Γ and \mathbb{Z}^2 .

If the Higgs field is strictly maximal so that $g_0 = g$, then there is no constant part. Hence, $A' \rightarrow Y'$ is Y' -isogenous to a product $E \times_{Y'} \cdots \times_{Y'} E$ with $E \rightarrow Y'$ modular.

Now let $C \rightarrow Y$ be a semistable family of curves of genus g with $J \rightarrow Y$ its corresponding family of Jacobians. Let $V \rightarrow U$ be the smooth part of $J \rightarrow Y$ and $S = Y - U$ the bad locus - as before with $A \rightarrow Y$ instead of $J \rightarrow Y$. Assume that $V \rightarrow U$ is a Shimura curve so that its Higgs field is maximal. If $Y = \mathbb{P}_{\mathbb{C}}^1$, then the Arakelov inequality above tells us that $S \neq \emptyset$. If we further assume that the Higgs field of $V \rightarrow U$ is not only maximal but strictly maximal, then the Jacobian has the following decomposition.

Corollary 2.3 (Structure of the Jacobian). *Let $C \rightarrow Y$ be a semistable family of curves of genus g whose Jacobian $J \rightarrow Y$ has a strictly maximal Higgs field. If $Y = \mathbb{P}_{\mathbb{C}}^1$, then $J \rightarrow Y$ is Y -isogenous to the g -fold product $E \times_Y \cdots \times_Y E$ of a modular family of elliptic curves $E \rightarrow Y$.*

Proof. Because of the Arakelov inequality we have $S \neq \emptyset$. So we may apply Theorem (2.2). Since $Y = \mathbb{P}_{\mathbb{C}}^1$ has no other étale coverings than automorphisms, the splitting takes place over Y . \square

We will show that for a fixed modular family $E \rightarrow Y$ there cannot exist families of curves $C \rightarrow Y$ of arbitrary large genus whose Jacobian is a g -fold product of $E \rightarrow Y$.

3. REDUCTION TO NUMBER FIELDS

We want to show that a family of curves $C \rightarrow Y$ whose Jacobian $J \rightarrow Y$ is Y -isogenous to the g -fold product of a modular family of elliptic curves $E \rightarrow Y$ is (after a base change) defined over a number field F which depends only on $E \rightarrow Y$ and not on g .

We say that a family of curves or group schemes $X \rightarrow Y$ is *defined over a number field F* if there is a curve Y_0/F and a family of curves or group schemes $X_0 \rightarrow Y_0$ over F such that $Y_0 \times_F \mathbb{C} \simeq Y$ over \mathbb{C} and the pull-back family $X_0 \times_F \mathbb{C} \rightarrow Y_0 \times_F \mathbb{C} \simeq Y$ is Y -birational to $X \rightarrow Y$, i.e. the families have isomorphic generic fibers.

We say that a Y -morphism f between two families $X \rightarrow Y$ and $Z \rightarrow Y$ is *defined over a number field F* if $X \rightarrow Y$ and $Z \rightarrow Y$ are defined over F and there is a Y_0 -morphism $f_0 : X_0 \rightarrow Y_0$ which coincides generically with the Y -morphism $f : X \rightarrow Y$ after the base change $\text{Spec } \mathbb{C} \rightarrow \text{Spec } F$.

As we will see, the reason why $C \rightarrow Y$ descends to a number field is that modular families of elliptic curves $E \rightarrow Y$ are defined over number fields. We start by describing the torsion structure of families of elliptic curves $E \rightarrow Y$ via Galois representations.

Let $K := \mathbb{C}(Y)$ be the function field of Y and K_v be the v -adic completion of K where v denotes a normalized discrete valuation of K induced by some point $y \in Y(\mathbb{C})$. Let $G_v = \text{Gal}(\overline{K_v}/K_v)$ be the absolute Galois group of K_v . By j_E we denote the j -invariant of an elliptic curve.

Proposition 3.1 (Galois action on torsion of Tate curves). *Let K_v be a v -adic complete field with residue field \mathbb{C} and absolute Galois group G_v , and let E/K_v be a Tate curve. Then for any prime power ℓ^n , we can find a basis (P_1, P_2) of $E[\ell^n](\overline{K_v})$ such that for any integer n' with $\ell^{n'+1} \nmid v(j_E)$, there is an element $\sigma \in G_v$ which acts on $E[\ell^n](\overline{K_v})$ with respect to the basis (P_1, P_2) like*

$$\begin{pmatrix} 1 & \ell^{n'} \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/\ell^n \mathbb{Z}).$$

In particular, for almost all prime powers ℓ^n there is a transvection, i.e. $n' = 0$.

Furthermore, the basis (P_1, P_2) can be chosen as follows: for P_1 we may take any ℓ^n -torsion point which specializes into the connected component of one, while for P_2 we may take any other point such that (P_1, P_2) forms a basis.

Proof. Mimic the proof of [Si94, V.6.1] using Tate's v -adic Uniformization Theorem. \square

From this we can conclude that the image of the action of the absolute Galois group $G = \text{Gal}(\overline{K}/K)$ acting on N -torsion points $E[N](\overline{K})$ of $E \rightarrow Y$ is huge.

Corollary 3.2 (Galois action on torsion points of E). *Let $E \rightarrow Y$ be a non-isotrivial family of elliptic curves and $K = \mathbb{C}(Y)$ the function field of Y . Then for any prime number ℓ there is a non-negative integer $n(\ell)$ such that for all prime powers ℓ^n , there are elements σ and σ' of G which act like*

$$\begin{pmatrix} 1 & \ell^{n(\ell)} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \ell^{n(\ell)} & 1 \end{pmatrix}$$

on $E[\ell^n](\overline{K})$ with respect to a suitable basis. Moreover, for almost all ℓ , we may choose $n(\ell) = 0$.

Proof. We argue as in [Ig59]. We may assume that $E \rightarrow Y$ has everywhere semistable reduction and a full level- ℓ^n -structure, i.e. there is an isomorphism of Y -group schemes $(\mathbb{Z}/\ell^n\mathbb{Z})_Y^2 \rightarrow E[\ell^n]$. This can always be achieved after a finite base change.

Choosing a point $y \in Y(\mathbb{C})$ such that $E \rightarrow Y$ has bad reduction in y , we find by Proposition (3.1) a basis (P_1, P_2) of $E[\ell^n](\overline{K})$ such that there is an element $\sigma \in G$ which acts with respect to (P_1, P_2) like

$$\begin{pmatrix} 1 & \ell^{n'} \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})$$

for n' with $\ell^{n'+1} \nmid v(j_E)$ where v is the valuation at y . Furthermore, P_1 specializes into the connected component of one while P_2 does not. So we may find another point $y' \in Y(\mathbb{C})$ such that P_2 will specialize into the connected component of one, since having a full level- ℓ^n -structure $E \rightarrow Y$ is the pull-back of the universal elliptic curve $E(\ell^n) \rightarrow X(\ell^n)$ parameterizing full level- ℓ^n -structures (we may assume that $\ell^n > 2$ because if the statement is true for ℓ^n , it is also true for ℓ^{n-1}). So using again (3.1) we will find an element $\sigma' \in G$ which acts with respect to the basis (P_2, P_1) like

$$\begin{pmatrix} 1 & \ell^{n''} \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})$$

for n'' with $\ell^{n''+1} \nmid v'(j_E)$ where v' is the valuation in y' . Of course upper triangle matrices with respect to (P_2, P_1) will be lower triangle matrices with respect to (P_1, P_2) . So choosing $n(\ell)$ such that $\ell^{n(\ell)+1} \nmid v(j_E)$ and $\ell^{n(\ell)+1} \nmid v'(j_E)$, we find two elements σ and σ' of $G_{\overline{K}/K}$ which act with respect to (P_1, P_2) like

$$\begin{pmatrix} 1 & \ell^{n(\ell)} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \ell^{n(\ell)} & 1 \end{pmatrix}.$$

In particular, $n(\ell)$ does only depend on ℓ or ℓ^n and for almost all ℓ , we may choose $n(\ell) = 0$ because $\ell \nmid v(j_E)$ and $\ell \nmid v'(j_E)$. \square

We can conclude the shape of endomorphisms on torsion groups of $E \rightarrow Y$.

Corollary 3.3 (Endomorphisms of torsion groups $E[N]$). *Let $E \rightarrow Y$ be a non-isotrivial family of elliptic curves. Then for any prime number ℓ , there is a non-negative integer $n(\ell)$ such that for all prime powers ℓ^n , every Y -endomorphisms of $E[\ell^n]$ is the sum of a multiplication-by- m map and a composition of the multiplication-by- $\ell^{n-n(\ell)}$ with an endomorphism of $E[\ell^{n(\ell)}]$.*

Moreover, for almost all ℓ we may choose $n(\ell) = 0$ so that the endomorphisms $\text{End}_Y(E[\ell^n])$ consist only of multiplication-by- m maps.

Proof. Any endomorphism of $E[\ell^n]$ is invariant under the action of Galois so that it has to lie in the center of the action of Galois on $E[\ell^n]$. In particular, an endomorphism must commute with the two matrices from Corollary (3.2). An elementary matrix calculation shows that any such endomorphism has to be represented by a matrix of the form

$$A = m \cdot I + \ell^{n-n(\ell)} \cdot M \in M_\ell(\mathbb{Z}/\ell^n\mathbb{Z})$$

where m is an integer, I is the identity matrix and M is some other matrix. The interpretation is the following. $m \cdot I$ corresponds to the multiplication-by- m map, while $\ell^{n-n(\ell)} \cdot M$ is the composition of the multiplication-by- $\ell^{n-n(\ell)}$ map $E[\ell^n] \rightarrow E[\ell^{n(\ell)}]$ with an endomorphism of $E[\ell^{n(\ell)}]$.

Moreover, Corollary (3.2) says that for almost all ℓ we have $n(\ell) = 0$ so that $\text{End}_Y(E[\ell^{n(\ell)}]) = \{0\}$. \square

This tells us that if the family $E \rightarrow Y$ is defined over some number field, then the same is true for the endomorphisms of $E[N]$.

Corollary 3.4 (Endomorphisms of torsion groups descend). *Let $E \rightarrow Y$ be a non-isotrivial family of elliptic curves defined over some number field F . Then there is a finite extension F' of F such that for all natural numbers N the Y -endomorphisms of $E[N]$ are defined over F' . In particular, the field F' depends only on $E \rightarrow Y$.*

Proof. It is enough to consider prime powers ℓ^n . By Corollary (3.3) for almost all ℓ the endomorphisms $\text{End}_Y(E[\ell^{n(\ell)}])$ consist only of multiplication-by- m maps which are clearly defined over F .

For the finitely many remaining ℓ , we also have to consider endomorphisms of $E[\ell^{n(\ell)}]$. Since these are finite in number, they will be defined over some finite extension F' of F depending only on $E \rightarrow Y$. \square

In particular, any family of Abelian varieties $A \rightarrow Y$ which is isogenous to a g -fold product of $E \rightarrow Y$ can be defined over the number field F' .

Proposition 3.5 (Isogenies and Abelian varieties descend). *Let $E \rightarrow Y$ a non-isotrivial family of elliptic curves defined over some number field F . Then there is a finite covering $Y' \rightarrow Y$ and a finite extension F' of F such that for every Y -isogeny h from any g -fold product of $E \rightarrow Y$ to any family of Abelian varieties $A \rightarrow Y$, the Y' -isogeny $h' = h \times_Y \text{id}_{Y'}$ is defined over F' . In particular, Y' and F' depend only on $E \rightarrow Y$ and not on g .*

Proof. Let H be the kernel of h . For a suitable number N , the group scheme H is contained in $E[N] \times_Y \cdots \times_Y E[N]$. After a suitable base change $Y' \rightarrow Y$, which may be chosen independently from H because of the finiteness result Corollary (3.3), we can describe H' as the kernel of an endomorphism of $E'[N] \times_{Y'} \cdots \times_{Y'} E'[N]$. Corollary (3.4) shows that H' will be defined over F' , i. e. H' is the pull-back of a subgroup scheme H'_0 of $E'_0[N] \times_{Y'_0} \cdots \times_{Y'_0} E'_0[N]$ with respect to the base change $\text{Spec } \mathbb{C} \rightarrow \text{Spec } F'$.

Now let $A'_0 \rightarrow Y'_0$ be the quotient of the g -fold product of $E'_0 \rightarrow Y'_0$ by H'_0 and let h'_0 be the quotient map. Clearly, h' and $A' \rightarrow Y'$ coincide generically with the pull-backs of h'_0 and A'_0 under the base change $\text{Spec } \mathbb{C} \rightarrow \text{Spec } F'$. Also Corollary (3.4) says that F' depends only on the family $E' \rightarrow Y'$ and, therefore, on $E \rightarrow Y$. \square

Moreover, we can say something about the isogenies which will occur.

Proposition 3.6 (Divisibility of the degree of isogenies). *Let $E \rightarrow Y$ be a non-isotrivial family of elliptic curves. Then there is a finite set of primes $S = S(E/Y)$, depending only on $E \rightarrow Y$, such that for every family of Abelian varieties $A \rightarrow Y$, which is Y -isogenous to a g -fold product of $E \rightarrow Y$, there is a Y -isogeny between the g -fold product of $E \rightarrow Y$ and $A \rightarrow Y$ whose degree has only prime divisors contained in S . In particular, the set S does not depend on g .*

Proof. Let $S = S(E/Y)$ be the set of primes ℓ such that the integers $n(\ell)$ from Corollary (3.2) are non-zero. Let $h : E \times_Y \cdots \times_Y E \rightarrow A$ be an isogeny and $H \subset \text{Ker}(h)$ a non-trivial simple subgroup scheme so that the order of G is some prime power ℓ^n .

Let $\ell \notin S$. If $g = 1$, then $H \simeq E[\ell]$ since $E[\ell]$ is irreducible by Corollary (3.3). Thus $h : E \rightarrow A$ factorizes through $E \simeq E/H \rightarrow A$. Proceeding like this, we can find an isogeny prime to all $\ell \notin S$.

If $g > 1$, then using Corollary (3.3) we see that H maps into a factor of $E \times_Y \cdots \times_Y E$ by a multiplication-by- m map. So after applying a suitable automorphism of $E \times_Y \cdots \times_Y E$, the subgroup H lies in a $g - 1$ -dimensional factor of the product. By induction we see that $(E \times_Y \cdots \times_Y E)/H$ is isomorphic to $E \times_Y \cdots \times_Y E$. \square

So far, we have seen that a family of Abelian varieties which is isogenous to a product of a modular family of elliptic curves can be defined over a number field. Now we want to have this result for a family of curves whose Jacobian is isogenous to a product of a modular family of elliptic curves.

Proposition 3.7 (Curves descend). *Let $E \rightarrow Y$ be a non-isotrivial family of elliptic curves defined over a number field F . Then there is a finite field extension F' of F and a curve Y' covering Y , both depending only on $E \rightarrow Y$, such that for any family of curves $C \rightarrow Y$, whose Jacobian $J \rightarrow Y$ is Y -isogenous to a g -fold product of $E \rightarrow Y$, there is finite covering $Y'' \rightarrow Y'$ of degree at most 2, such that $C \times_Y Y'' \rightarrow Y''$ is defined over F' . In particular, Y' and the degree of $Y'' \rightarrow Y'$ do not depend on g .*

Proof. Let $S = S(E/Y)$ be the set of primes from Proposition (3.6) about the divisibility of the degree of isogenies. Let $N \geq 3$ be an integer which is not divisible by any prime in S . (The choice of N will later ensure that the Jacobian is equipped with a level- N -structure, see below.)

After a finite extension $Y' \rightarrow Y$, we may assume that $E' \rightarrow Y'$ is equipped with a level- N -structure α' . By further extending $Y' \rightarrow Y$ we may also assume that the conclusion of Proposition (3.5) about descending isogenies and Abelian varieties holds. The extension $Y' \rightarrow Y$ depends only on $E \rightarrow Y$.

Now let $C \rightarrow Y$ be a curve whose Jacobian $J \rightarrow Y$ is Y -isogenous to a g -fold product of $E \rightarrow Y$. We may assume by Proposition (3.6) that there is a Y -isogeny h from $E \times_Y \cdots \times_Y E$ to J whose degree has only prime divisors in S . Thus, the level- N -structure α' will be mapped under h' injectively into $J' \rightarrow Y'$ so that $J' \rightarrow Y'$ itself is equipped with a level- N -structure which we will call β' .

By Proposition (3.5) and the choice of Y' , we see that $J' \rightarrow Y'$ together with its principal polarization θ' (which is just a special kind of isogeny) are defined over F' . Also the level- N -structure β' on $J' \rightarrow Y'$ is defined over F' since it is the image of the level- N -structure α' of $E' \times_{Y'} \cdots \times_{Y'} E' \rightarrow Y'$ under h' . So, the triple $(J' \rightarrow Y', \theta', \beta')$ is defined over F' .

The canonical morphism between fine moduli spaces

$$j^{(N)} : \mathcal{M}_g^{(N)} \rightarrow \mathcal{A}_{g,1}^{(N)}$$

which sends a curve with level- N -structure to its principally polarized Jacobian with the same level- N -structure is 2-to-1 over its image for $g \geq 3$ respectively 1-to-1 for $g = 2$, see [OS80]. Hence, the curve $C' \rightarrow Y'$ can be defined over F' after applying a base change $Y'' \rightarrow Y'$ of degree at most 2.

As mentioned above, the choice of Y' and F' depend only on the family of elliptic curves $E \rightarrow Y$. \square

So we see that a family of curves, whose Jacobian is isogenous to the g -fold product of a modular family of curves, can be defined over a number field after a finite base change of degree at most 2. And the number field does not depend on the given family of curves but only on the modular family of elliptic curves.

4. REDUCTION TO FINITE FIELDS

Let $C \rightarrow Y$ be a family of curves, i.e. a flat, projective morphism whose generic fiber is smooth and geometrically connected. Assume that the Jacobian $J \rightarrow Y$ is Y -isogenous to the g -fold product $E \times_Y \cdots \times_Y E$ of a modular family of elliptic curves $E \rightarrow Y$.

Because of Theorem (3.7) we may assume that our base field is a number field F . We want to show that there is a finite prime of F such that the family $\tilde{C} \rightarrow \tilde{Y}$ obtained by reduction modulo this prime has a also a smooth generic fiber. It is clearly possible to find such a prime depending on the given family $C \rightarrow Y$. But we want to show that there is a choice of this prime which depends only on $E \rightarrow Y$ and not on $C \rightarrow Y$ or g , so that for any family $C \rightarrow Y$ of arbitrarily large genus g , whose Jacobian is Y -isogenous to $E \times_Y \cdots \times_Y E$, its reduction $\tilde{C} \rightarrow \tilde{Y}$ will still be generically smooth.

Therefor we need a criterion for the generic smoothness of a family of curves $C \rightarrow Y$ after reduction modulo a prime. The characterization is given in terms of the existence of certain endomorphisms on the principally polarized Jacobian $J \rightarrow Y$ of $C \rightarrow Y$.

In this section we consider all schemes, morphisms and fiber products to live over an at most 1-dimensional base scheme S which is suppressed from the notation.

Definition 4.1 (Split principally polarized Abelian variety). A principally polarized Abelian variety (A, λ_A) *splits* if there are two positive-dimensional principally polarized Abelian varieties (B, λ_B) and (C, λ_C) such that (A, λ_A) is isomorphic to $(B \times C, \lambda_B \times \lambda_C)$ as a principally polarized Abelian variety.

For the next proposition, we assume that S is the spectrum of an algebraically closed field of arbitrary characteristic.

Note that a smooth curve C has a proper Jacobian J . But the converse is not true, since e.g. two smooth curves intersecting transversally in one point also have a proper Jacobian. So we need a criterion to distinguish between these cases.

Proposition 4.2 (Reducibility criterion for curves). *Let C be a curve with proper Jacobian (J, λ) . Then C is reducible if and only if (J, λ) splits as a principally polarized Abelian variety.*

Proof. If C has a proper Jacobian then it is either smooth or it consists of smooth irreducible components C_i intersecting in a way such that they form a tree. Hence, if C is reducible, then (J, λ) is the product of the Jacobians (J_i, λ_i) of the smooth components C_i .

It remains to show the converse that C is reducible if (J, λ) splits. Assume that $(J, \lambda) = (A_1 \times A_2, \lambda_1 \times \lambda_2)$ where (A_i, λ_i) are positive-dimensional principally polarized Abelian varieties and that C is smooth. Choose an embedding $C \xrightarrow{f} J$ and let p_i be the projection $A_1 \times A_2 \xrightarrow{p_i} A_i$.

Then $C_i := p_{i*}f(C) \subset A_i$ is a 1-cycle generating A_i (i. e. A_i is the smallest Abelian subvariety of A_i containing C_i) because C generates its Jacobian $J = A_1 \times A_2$. Define $\tilde{C} := C_1 \times \{0\} + \{0\} \times C_2 \subset A_1 \times A_2 = J$. It follows that \tilde{C} is a 1-cycle generating J .

Let $\Theta_i \subset A_i$ be a divisor inducing the polarization $A_i \xrightarrow{\lambda_i} \widehat{A_i}$. Then the divisor $\Theta := \Theta_1 \times A_2 + A_1 \times \Theta_2$ on $A_1 \times A_2 = J$ induces the polarization $J \xrightarrow{\lambda} \widehat{J}$. We compute the intersection number

$$\begin{aligned} (\tilde{C} \cdot \Theta) &= (C_1 \times \{0\} \cdot \Theta_1 \times A_2) + (C_1 \times \{0\} \cdot A_1 \times \Theta_2) \\ &\quad + (\{0\} \times C_2 \cdot \Theta_1 \times A_2) + (\{0\} \times C_2 \cdot A_1 \times \Theta_2). \end{aligned}$$

The two middle terms are zero as an application of the projection formula shows. An other application of the projection formula on the remaining two terms gives us

$$(\tilde{C} \cdot \Theta) = (f(C) \cdot \Theta_1 \times A_2) + (f(C) \cdot A_1 \times \Theta_2) = (f(C) \cdot \Theta) = g$$

where the last equality follows from the fact that C is the Jacobian of C .

From the Matsusaka-Ran Theorem [Co84] follows that the (A_i, λ_i) are the Jacobians of the curves C_i and that the C_i are components of C , so that C has to be reducible. \square

A characterization of split principally polarized Abelian variety (A, λ) is given in the next proposition. There, the *Rosati-involution* on $\text{End}_S(A)$ is denoted by

$$f \mapsto f^\dagger := \lambda^{-1} \circ \widehat{f} \circ \lambda$$

where $\widehat{A} \xrightarrow{\widehat{f}} \widehat{A}$ is the dual map of f .

Proposition 4.3 (Splitting criterion). *For a principally polarized Abelian variety (A, λ_A) , the following two statements are equivalent:*

- (i) (A, λ_A) splits, i. e. (A, λ_A) is isomorphic as a principally polarized Abelian variety to a product $(B \times C, \lambda_B \times \lambda_C)$ of two positive-dimensional principally polarized Abelian varieties (B, λ_B) and (C, λ_C) .
- (ii) (A, λ_A) possesses a non-trivial symmetric idempotent endomorphism, i. e. it exists a map $f \in \text{End}_S(A)$ different from the identity and the zero map such that the two relations $f^\dagger = f$ and $f^2 = f$ hold.

Proof. (i) \Rightarrow (ii) Let $h : A \xrightarrow{\sim} B \times C$ be an isomorphism of principally polarized Abelian varieties and define f to be the following composition of maps

$$\begin{array}{ccc} B \times C & \xrightarrow{1 \times 0} & B \times C \\ \uparrow h & & \downarrow h^{-1} \\ A & \xrightarrow{=:f} & A. \end{array}$$

Then f is an idempotent and symmetric endomorphism of A . For the idempotence consider the commutative diagram

$$\begin{array}{ccccc}
 & & 1 \times 0 & & \\
 & \searrow & & \nearrow & \\
 B \times C & \xrightarrow{1 \times 0} & B \times C & \xrightarrow{1 \times 0} & B \times C \\
 \uparrow h & & \downarrow h^{-1} & \uparrow h & \downarrow h^{-1} \\
 A & \xrightarrow{f} & A & \xrightarrow{f} & A \\
 & \searrow & & \nearrow & \\
 & & f^2 \stackrel{!}{=} f & &
 \end{array}$$

where the lower row gives us f^2 . If we follow the upper way around we get f . So f and f^2 coincide and, therefore, f is idempotent. For the symmetry look at the diagram

$$\begin{array}{ccccccc}
 & & \hat{f} & & & & \\
 & \searrow & & \nearrow & & & \\
 \hat{A} & \xrightarrow{\hat{h}^{-1}} & \hat{B} \times \hat{C} & \xrightarrow{\hat{1} \times \hat{0}} & \hat{B} \times \hat{C} & \xrightarrow{\hat{h}} & A \\
 \uparrow \lambda_A & & \uparrow \lambda_B \times \lambda_C & & \downarrow \lambda_B^{-1} \times \lambda_C^{-1} & & \downarrow \lambda_A^{-1} \\
 A & \xrightarrow{h} & B \times C & \xrightarrow{1 \times 0} & B \times C & \xrightarrow{h^{-1}} & A \\
 & \searrow & & \nearrow & & & \\
 & & f \stackrel{!}{=} f^\dagger & & & &
 \end{array}$$

which is commutative since dualizing endomorphisms commutes with inverting them. Again the lower row gives us f while the upper way around we obtain f^\dagger . So f and f^\dagger are identical, telling us that f is a symmetric and idempotent endomorphism of A .

(ii) \Rightarrow (i) Let $A \xrightarrow{f} A$ be a symmetric idempotent endomorphism of A . Define $B := \text{Im}(A \xrightarrow{f} A)$ and $C := \text{Im}(A \xrightarrow{1-f} A)$. Then we get a homomorphism

$$B \times C = fA \times (1-f)A \xrightarrow{h} A, \quad (b, c) \mapsto b + c.$$

Since f is idempotent, the homomorphism

$$A \longrightarrow B \times C = fA \times (1-f)A, \quad a \mapsto (fa, (1-f)a)$$

is an inverse map for h and, therefore, $B \times C \xrightarrow{h} A$ is an isomorphism of Abelian varieties.

Let λ_B and λ_C be the restrictions of λ_A on B and C . This makes B and C into principally polarized Abelian varieties. Since f is symmetric the two

diagrams

$$\begin{array}{ccc}
 \widehat{A} & \xrightarrow{\widehat{f}} & \widehat{A} \\
 \lambda_A \uparrow & & \uparrow \lambda_A \\
 A & \xrightarrow{f} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 \widehat{A} & \xrightarrow{\widehat{1-f}} & \widehat{A} \\
 \lambda_A \uparrow & & \uparrow \lambda_A \\
 A & \xrightarrow{1-f} & A
 \end{array}$$

commute. In particular, B , which is the image of f , is mapped under λ_A into the image of \widehat{A} under \widehat{f} . The same holds for C and $1 - f$. But then also the diagram

$$\begin{array}{ccc}
 \widehat{B} \times \widehat{C} = \widehat{f}\widehat{A} \times (\widehat{1-f})\widehat{A} & \xleftarrow{\widehat{h}} & \widehat{A} \\
 \lambda_B \times \lambda_C \uparrow & & \uparrow \lambda_A \\
 B \times C = fA \times (1-f)A & \xrightarrow{h} & A
 \end{array}$$

commutes. Hence, as a principally polarized Abelian variety (A, λ_A) is isomorphic to $(B \times C, \lambda_B \times \lambda_C)$ via the map h . \square

Assume that an Abelian scheme splits after reduction modulo a prime so that it owns a symmetric idempotent endomorphism. To show that it is already split before reduction we want to lift the endomorphism.

Let $S = \text{Spec } R$ be the spectrum of a henselian discrete valuation ring with quotient field K and residue field k . We use the following notational convention. A small subscript denotes the base scheme. So the schemes X_K and X_k are schemes over $\text{Spec } K$ and $\text{Spec } k$, respectively. A Scheme over S is simply denoted by X instead of X_S . Then X_K is its general fiber and X_k is its special fiber. Let $A \rightarrow S$ be an Abelian scheme. The question we will study is when does an endomorphism of A_k lift to an endomorphism of A_K .

Definition 4.4 (The lifting property). We say that every endomorphism of A_k *lifts* if the restriction map

$$\begin{array}{ccc}
 \text{End}_S(A) & \longrightarrow & \text{End}_k(A_k) \\
 f & \longmapsto & f_k := f|_{A_k}
 \end{array}$$

is an isomorphism.

Example 4.5. If $E \rightarrow S$ is a relative elliptic curve such that E_K and E_k are both elliptic curves without complex multiplication so that the endomorphism rings $\text{End}_S(E)$ and $\text{End}_k(E_k)$ are isomorphic to \mathbb{Z} , then the restriction map is an isomorphism because the endomorphisms are the multiplication-by- m maps.

The same is true for the g -fold product of the elliptic curve E since in this case any endomorphism is build up from multiplication-by- m maps which lift.

The lifting property is invariant under étale isogenies.

Proposition 4.6 (The lifting property and étale isogenies). *Let A and B be two principally polarized Abelian schemes over S and $h : A \rightarrow B$ an étale S -isogeny. If every endomorphism of A_k lifts, then every endomorphism of B_k lifts too.*

Proof. Look at the commutative diagram

$$\begin{array}{ccc} \mathrm{End}_S(B) & \xrightarrow{h^*} & \mathrm{End}_S(A) \\ \downarrow & & \downarrow \cong \\ \mathrm{End}_k(B_k) & \xrightarrow{h_k^*} & \mathrm{End}_k(A_k) \end{array}$$

where h^* is given by $f \mapsto h^*f := h^\dagger \circ f \circ h$ and h_k^* is the analogous map for endomorphisms of the special fiber. Let $f_k \in \mathrm{End}_k(B_k)$ be an endomorphism of B_k . We want to lift f_k to an endomorphism $f \in \mathrm{End}_S(B)$ so that $f|_{B_k} = f_k$. Look at the map $h_k^*f_k \in \mathrm{End}_k(A_k)$. Since A has the lifting property, the map $h_k^*f_k$ lifts to a map $A \xrightarrow{u} A$ so that $u_k = h_k^*f_k$. If we can show that $A \xrightarrow{u} A$ lies in the image of h^* , i.e. there is a map $f \in \mathrm{End}_S(B)$ with $h^*f = h^\dagger \circ f \circ h = u$, then the map f is a lifting of f_k because of the commutativity of the diagram above.

We know that $u_k = h_k^*f_k = h_k^\dagger \circ f_k \circ h_k$ factorizes through h_k so that $\mathrm{Ker}(h_k)$ is a subgroup scheme of $\mathrm{Ker}(u_k)$. Since our base S is henselian and $\mathrm{Ker}(h_k)$ étale, there is a subgroup scheme $G \subset \mathrm{Ker}(u)$ such that $G_k = \mathrm{Ker}(h_k)$. But then, being a subgroup scheme of A , the group scheme G has to coincide with $\mathrm{Ker}(h)$ since finite étale schemes over S are uniquely determined by their special fiber [Mi80, p.34]. Hence, u factorizes through h , i.e. there is a map $B \xrightarrow{g} A$ such that $u = g \circ h$ holds.

Analogously one shows that the dual \hat{g} of g factorizes through the dual \hat{h}^\dagger of h^\dagger . Hence, there exists an endomorphism $B \xrightarrow{f} B$ such that $g = h^\dagger \circ f$ is valid. Therefore, we get the identity $u = h^*f$ and f becomes a lifting of f_k . This implies that the Abelian scheme B also has the lifting property. \square

In particular, Abelian schemes isogenous to a g -fold product of a relative elliptic curve without complex multiplication have the lifting property.

Let us return to our situation in the beginning of this section. Let Y_F be a smooth, projective, geometrically connected curve over some number field F . Let \mathcal{O}_F denote the ring of integers of F and let $\mathrm{Spec} \mathbb{F}_q \rightarrow \mathrm{Spec} \mathcal{O}_F$ be any finite point of $\mathrm{Spec} \mathcal{O}_F$. We can extend $Y_F \rightarrow \mathrm{Spec} F$ to an arithmetic minimal model $Y \rightarrow \mathrm{Spec} \mathcal{O}_F$, i.e. $Y \rightarrow \mathrm{Spec} \mathcal{O}_F$ is an integral, proper, regular, excellent and flat surface of finite type with general fiber Y_F together with the usual minimality property similar to the geometric case, see [Ch86]. Let $Y_{\mathbb{F}_q} \rightarrow \mathrm{Spec} \mathbb{F}_q$ be the special fiber of $Y \rightarrow \mathrm{Spec} \mathcal{O}_F$ over the point $\mathrm{Spec} \mathbb{F}_q \rightarrow \mathrm{Spec} \mathcal{O}_F$.

Let K be the function field of Y_F so that $\text{Spec } K \rightarrow Y_F$ is the generic point. Let k be the function field of an irreducible component of $Y_{\mathbb{F}_q}$ so that $\text{Spec } k \rightarrow Y_{\mathbb{F}_q}$ is the generic point of the corresponding irreducible component. Furthermore, let R be the local ring of Y at this irreducible component. In particular, R is a discrete valuation ring (Y is regular) with generic point $\text{Spec } K \rightarrow \text{Spec } R$ and special point $\text{Spec } k \rightarrow \text{Spec } R$.

$$\begin{array}{ccccc}
\{C_K, J_K, E_K\} & \hookrightarrow & \{C, J, E\} & \longleftarrow & \{C_k, J_k, E_k\} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Spec } K & \hookrightarrow & \text{Spec } R & \longleftarrow & \text{Spec } k \\
\downarrow & & \downarrow & & \downarrow \\
Y_F & \hookrightarrow & Y & \longleftarrow & Y_{\mathbb{F}_q} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Spec } F & \hookrightarrow & \text{Spec } \mathcal{O}_F & \longleftarrow & \text{Spec } \mathbb{F}_q
\end{array}$$

Finally, let C_K be a smooth, projective, geometrically connected curve defined over the function field K , J_K its Jacobian and E_K an elliptic curve. We may extend $C_K \rightarrow \text{Spec } K$ to a minimal model $C \rightarrow \text{Spec } R$ with Jacobian $J \rightarrow \text{Spec } R$ and we denote the special fibers of these models by $C_k \rightarrow \text{Spec } k$ and $J_k \rightarrow \text{Spec } k$. Of course, also $E_K \rightarrow \text{Spec } K$ extends to a (Néron) model $E \rightarrow \text{Spec } R$ with special fiber $E_k \rightarrow \text{Spec } k$.

With a view towards Proposition (3.7), we want to allow our curve C to be defined over some finite field extension K' of K rather than over K itself. Therefore, let $Y' \rightarrow Y$ be the map of minimal arithmetic surfaces induced by $\text{Spec } K' \rightarrow \text{Spec } K$, so that K' is the function field of Y' . Let $Y'_{\mathbb{F}_q}$ be the fiber of $Y' \rightarrow \text{Spec } \mathcal{O}_F$ over the point $\text{Spec } \mathbb{F}_q \rightarrow \text{Spec } \mathcal{O}_F$, k' the function field of an irreducible component of $Y'_{\mathbb{F}_q}$ and R' the local ring of Y' at this irreducible component. We can extend $C_{K'}$ to a model $C' \rightarrow \text{Spec } R'$ and denote its special fiber by $C_{k'}$.

Theorem 4.7 (Reduction to finite fields). *Let E_K be a non-isotrivial elliptic curve. Then there is a finite point $\text{Spec } \mathbb{F}_q \rightarrow \text{Spec } \mathcal{O}_F$, depending only on E_K , such that the following property holds:*

Let J_K be an Abelian variety which is K -isogenous to a g -fold product of E_K . Let K' be a finite extension of K such that $J_{K'}$ becomes the Jacobian of a projective and geometrically connected curve $C_{K'}$. Then $C_{K'}$ is smooth over K' if and only if its reduction $C_{k'}$ with respect to $\text{Spec } \mathbb{F}_q \rightarrow \text{Spec } \mathcal{O}_F$ is smooth over k' .

Proof. First, we start with the case $K' = K$. We choose a finite point $\text{Spec } \mathbb{F}_q \rightarrow \text{Spec } \mathcal{O}_F$ of residue characteristic p such that the following properties are fulfilled.

- (1) Y_F has good reduction at $\text{Spec } \mathbb{F}_q$, i. e. the fiber $Y_{\mathbb{F}_q}$ is a smooth curve. This depends only on K – the function field of Y_F – and is true for almost all points of $\text{Spec } \mathcal{O}_F$.
- (2) $E_K \rightarrow \text{Spec } K$ extends to a smooth proper model $E \rightarrow \text{Spec } R$ such that $E_k \rightarrow \text{Spec } k$ is a non-isotrivial elliptic curve. This is true for almost all points of $\text{Spec } \mathcal{O}_F$ and depends only on E_K .
- (3) There is an isogeny $E_K \times_K \cdots \times_K E_K \rightarrow J_K$ such that its degree is prime to p . Using Proposition (3.6) we see that this is true for almost all points of $\text{Spec } \mathcal{O}_F$ and depends only on E_K . Together with (2) this property will enable us to lift endomorphisms of J_k to endomorphisms of J_K with the help of Proposition (4.6).

Since the three conditions above hold separately for all but finitely many points of $\text{Spec } \mathcal{O}_F$, we can find a point $\text{Spec } \mathbb{F}_q \rightarrow \text{Spec } \mathcal{O}_F$ fulfilling all conditions simultaneously. The choice of this point depends only on E_K .

As explained above, let R be the local ring of Y at $Y_{\mathbb{F}_q}$. Extend the curve $C_K \rightarrow \text{Spec } K$ to a minimal model $C \rightarrow \text{Spec } R$. Its Jacobian $J \rightarrow \text{Spec } R$ is equipped with a canonical principal polarization $\lambda : J \rightarrow \hat{J}$ such that (J_K, λ_K) and (J_k, λ_k) are the principally polarized Jacobian of C_K and C_k , respectively. Since by assumption J_K is K -isogenous to the g -fold product of E_K , the Jacobian J_k is k -isogenous to the g -fold product of E_k (actually $J \rightarrow \text{Spec } R$ is isogenous over $\text{Spec } R$ to the g -fold product of $E \rightarrow \text{Spec } R$).

Let \hat{R} be the completion of R and \hat{K} its quotient field. So after the base change $\text{Spec } \hat{R} \rightarrow \text{Spec } R$, we get a model $C_{\hat{R}} \rightarrow \text{Spec } \hat{R}$ with generic fiber $C_{\hat{K}} \rightarrow \text{Spec } \hat{K}$ and special fiber $C_k \rightarrow \text{Spec } k$.

As discussed in the proof of Proposition (4.2), a non-smooth curve with proper Jacobian becomes reducible after some finite base change. By Proposition (4.2), the reducibility of the curve is equivalent to the splitting of its Jacobian. By Proposition (4.3), the splitting is equivalent to the existence of a symmetric idempotent endomorphism of the Jacobian.

Since Proposition (4.6) tells us that $J_{\hat{K}}$ has the lifting property, a symmetric idempotent endomorphism exists on $J_{\hat{K}}$ if and only if it exists on J_k (apply a finite base change $\text{Spec } \hat{S} \rightarrow \text{Spec } \hat{R}$ if necessary). Hence C_K is smooth over K if and only if C_k is smooth over k .

Now, if K' is a finite extension of K , with the same choice of prime $\text{Spec } \mathbb{F}_q \rightarrow \text{Spec } \mathcal{O}_F$, as above, the Jacobian $J_{\hat{K}'}$ still has the lifting property. So, replacing K, R, k by K', R', k' , respectively, verbatim the same argument, as before, shows that $C_{K'}$ is smooth over K' if and only if $C_{k'}$ is smooth over k' . \square

In particular, taking the minimal model \tilde{C} of C_k over $\tilde{Y} = Y_{\mathbb{F}_q}$, the generically smooth family of curves $C \rightarrow Y$ induces a corresponding generically smooth family of curves $\tilde{C} \rightarrow \tilde{Y}$ over a suitable finite field \mathbb{F}_q , which does not depend on $C \rightarrow Y$. In the next section, we see that the genus g of the fibers of $\tilde{C} \rightarrow \tilde{Y}$, which is the same as the genus of the fibers of $C \rightarrow Y$, is bounded.

5. BOUNDING THE GENUS IN CHARACTERISTIC p

Let \mathbb{F}_q be a finite field of characteristic p . Serre has shown in [Se97] based on the work of Tsfasman and Vlăduț [TV97] that for a curve C/\mathbb{F}_q , whose Jacobian J/\mathbb{F}_q is \mathbb{F}_q -isogenous to a product of elliptic curves, the genus g is bounded.

We will compute an explicit bound for an easy special case and generalize the statement for families of curves.

Proposition 5.1 (Explicit bound). *Let C/\mathbb{F}_q be a curve of genus g whose Jacobian J/\mathbb{F}_q is \mathbb{F}_q -isogenous to the g -fold product of an elliptic curve E/\mathbb{F}_q . If $\text{Tr}(F_E | H^1(\bar{E}, \mathbb{Q}_\ell)) > 0$, then $g \leq q + 1$.*

Proof. Isogenies become isomorphisms on ℓ -adic cohomology, so that we have isomorphisms (where $\bar{C} = C \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$, etc.)

$$H^1(\bar{C}, \mathbb{Q}_\ell) \cong H^1(\bar{J}, \mathbb{Q}_\ell) \cong H^1(\bar{E} \times \cdots \times \bar{E}, \mathbb{Q}_\ell) \cong \bigoplus_{i=1}^g H^1(\bar{E}, \mathbb{Q}_\ell)$$

compatible with the action of Galois. Therefore, the Weil conjectures tell us that the number of \mathbb{F}_q -rational points of C is given by

$$\begin{aligned} 0 \leq \#C(\mathbb{F}_q) &= q + 1 - \text{Tr}(F_C | H^1(\bar{C}, \mathbb{Q}_\ell)) \\ &= q + 1 - g \cdot \text{Tr}(F_E | H^1(\bar{E}, \mathbb{Q}_\ell)) \end{aligned}$$

where F_C and F_E denote the q -th power Frobenius on C and E , respectively. Hence, it follows that

$$g \leq \frac{q + 1}{\text{Tr}(F_E | H^1(\bar{E}, \mathbb{Q}_\ell))} \leq q + 1$$

giving the desired bound. \square

With the same method, one can compute explicit bounds for other Frobenius traces, too, but we don't need to.

Now, let Y be a smooth, projective, geometrically connected curve over some finite field \mathbb{F}_q and $C \rightarrow Y$ a semistable family of curves of genus g . To bound the genus g , we will count the minimal number δ of singularities in the geometric fibers of $C \rightarrow Y$ and compare it with the following natural upper bound.

Proposition 5.2 (Upper bound for δ). *Let $f : C \rightarrow Y$ be a semistable family of curves of genus $g \geq 2$. Then*

$$\delta \leq 12 \cdot \deg f_* \omega_{C/Y}$$

where δ is the number of singularities in the geometric fibers of f and $\omega_{C/Y}$ is the relative dualizing sheaf.

Proof. The characteristic p case was proven by Szpiro in [Sz78, Prop.1]. \square

If the Jacobian $J \rightarrow Y$ of $C \rightarrow Y$ is Y -isogenous to the g -fold product of a non-isotrivial family of elliptic curves $E \rightarrow Y$, then we can express this upper bound in terms of the genus g and the height of $E \rightarrow Y$.

Recall that the *height* of a group scheme $G \rightarrow Y$ is given by

$$h(G) := \deg(s^* \Omega_{G/Y}^1)$$

where $s : Y \rightarrow G$ is the zero section.

Corollary 5.3 (Bounding δ for decomposable Jacobians). *Let $C \rightarrow Y$ be a semistable family of curves whose Jacobian $J \rightarrow Y$ is Y -isogenous to the g -fold product of a non-isotrivial family of elliptic curves $E \rightarrow Y$. Then*

$$\delta \leq 12 h(E) \cdot g$$

where $h(E)$ is the height of $E \rightarrow Y$. In particular, the constant $h(E)$ depends only on $E \rightarrow Y$, so that δ is linearly bounded by g .

Proof. Let $E \times_Y \cdots \times_Y E \rightarrow J$ be an isogeny with kernel N . Then we have $h(J) = h(E \times_Y \cdots \times_Y E) - h(N)$. Since E is non-isotrivial, N is an extension of an étale group scheme by some factors of the form μ_{p^n} . Both group schemes have height zero (the Cartier dual of μ_{p^n} is étale), so that $h(N) = 0$.

Furthermore, we have $h(E \times_Y \cdots \times_Y E) = g \cdot h(E)$ since the sheaf $\Omega_{E \times \cdots \times E/Y}^1$ is isomorphic to $\bigoplus_{i=1}^g p_i^* \Omega_{E/Y}^1$ where p_i is the projection on the i -th factor.

The height of the Jacobian is related to $C \rightarrow Y$ by $h(J) = \deg f_* \omega_{C/Y}$ because $\det s^* \Omega_{J/Y}^1 \cong \det f_* \omega_{C/Y}$ [Fa83, p.351]. So we have

$$\delta \leq 12 \cdot \deg f_* \omega_{C/Y} = 12 \cdot h(J) = 12 h(E) \cdot g$$

where the inequality is given by Proposition (5.2). \square

This gives us an upper bound for δ in terms of g . For the lower bound we will use Proposition (5.1). To determine how many fibers of $E \rightarrow Y$ have a positive Frobenius trace, we will use the Sato-Tate conjecture.

Theorem 5.4 (Sato-Tate Conjecture). *Let $E \rightarrow Y$ be a non-isotrivial family of elliptic curves and a and b two real numbers between 0 and π . Then*

$$\lim_{n \rightarrow \infty} \frac{\#\{y \in Y(\mathbb{F}_{q^n}) \mid a \leq \Theta(y) \leq b\}}{q^n} = \frac{2}{\pi} \int_a^b \sin^2 \varphi \, d\varphi$$

where $\Theta(y)$ is the angle of a Frobenius eigenvalue of the fiber E_y , i. e. the eigenvalues of the Frobenius acting on $H^1(\bar{E}_y, \mathbb{Q}_\ell)$ are given by $q^{n/2} \cdot e^{\pm \Theta(y)i}$.

Proof. This was proven by Deligne in [De80, p.212, (3.5.7)]. \square

We derive the following lower bound for the number of singularities δ . Since later we want to apply this Proposition together with Proposition (3.7) we allow the families of curves to be defined over some finite covering $Y' \rightarrow Y$.

Proposition 5.5 (Lower bound for δ). *Let $E \rightarrow Y$ be a non-isotrivial family of elliptic curves and $Y' \rightarrow Y$ some finite covering of degree d . Let $C' \rightarrow Y'$ be a semistable family of curves of genus g whose Jacobian $J' \rightarrow Y'$ is Y' -isogenous to the g -fold product of $E' \rightarrow Y'$. Then there is a constant $c = c(E/Y, d) > 0$, depending only on $E \rightarrow Y$ and d such that*

$$c \cdot \frac{\log g}{\log \log g} \cdot g \leq \delta$$

where δ is the number of singularities in the geometric fibers of $C' \rightarrow Y'$. In particular, δ is not linearly bounded above by g and the lower bound does not depend on the particular choice of the covering $Y' \rightarrow Y$.

Proof. We consider the case $d = 1$ first. After enlarging q if necessary, using the Sato-Tate-conjecture, we may assume that

$$\#\left\{y \in Y(\mathbb{F}_{q^n}) \mid 0 \leq \Theta(y) < \frac{\pi}{2}\right\} > \frac{1}{4}q^n.$$

If $g > q^n + 1$, then a fiber over an \mathbb{F}_{q^n} -rational point y of Y with $\Theta(y) < \frac{\pi}{2}$ has to be singular by (5.1). Its Jacobian is either isogenous to the g -fold product of a single elliptic curve or a torus. In the toric case, the curve has at least g singularities. In the compact case, the curve is a chain of smooth curves each of genus less or equal to $q^n + 1$. Such a curve will have at least $\lfloor \frac{g}{q^n+2} \rfloor$ singularities. Underestimating the number of singularities, we can say that in any case we have at least $\frac{g}{2q^n}$ singularities. So the total number of singularities we get from these fibers is at least

$$\frac{1}{4}q^n \cdot \frac{g}{2q^n} = \frac{1}{8}g$$

singularities.

There is one point we have to take care of. If m is a natural number dividing n , then $Y(\mathbb{F}_{q^m}) \subset Y(\mathbb{F}_{q^n})$. So saying that we get $\frac{1}{8}g$ singularities from the \mathbb{F}_{q^m} -rational fibers and additional $\frac{1}{8}g$ singularities from the \mathbb{F}_{q^n} -rational fibers is not fully correct because we possibly count some singularities more than once. To deal with this problem we will only consider extensions \mathbb{F}_{q^e} of prime degree e .

Hence assume that $g - 1 > q^2, q^3, q^5, q^7, q^{11}, \dots, q^e, \dots$ where the exponents e are prime numbers. How many $q^e < g - 1$ with e prime are there? It is the number of primes e with $e < \log_q(g - 1)$. So by the prime number theorem, there is a constant $c_1 > 0$ such that there are at least $c_1 \frac{\log_q(g-1)}{\log \log_q(g-1)}$ such primes e (c_1 is a little bit less than 1 if g is large). So we get not less than

$$c_1 \frac{\log_q(g-1)}{\log \log_q(g-1)} \cdot \frac{1}{8}g$$

singularities up to multiply counted ones.

Thus, we have to deal with the singularities we counted more than once, namely the ones coming from fibers defined over \mathbb{F}_q -rational points because $Y(\mathbb{F}_q) \subset Y(\mathbb{F}_{q^n})$ for all n . Using a bad estimate for $\#Y(\mathbb{F}_q)$, we assume that there are at most $2q$ \mathbb{F}_q -rational points (q not too small, enlarge if necessary). Then we counted at most $2q \cdot \frac{g}{2q^e} = \frac{g}{q^{e-1}}$ points too often for each prime e . So an upper bound for the total error is

$$\sum_{e \text{ prime}} \frac{g}{q^{e-1}} \leq 2g.$$

Therefore, the corrected total number of singularities we counted is

$$\left(\frac{c_1}{8} \frac{\log_q(g-1)}{\log \log_q(g-1)} - 2 \right) g.$$

Remember that we enlarged q to apply the Sato-Tate-conjecture for $E \rightarrow Y$. So for our original q , we can say that there is a constant $c > 0$ depending only on $E \rightarrow Y$ such that there are at least $c \frac{\log g}{\log \log g} g$ singularities.

Now assume that we have a covering $Y' \rightarrow Y$ of degree $d \geq 1$. Then any \mathbb{F}_{q^n} -rational point of Y has at least one $\mathbb{F}_{q^{rn}}$ -rational preimage with $r \leq d$. So applying the Sato-Tate-conjecture on $E' \rightarrow Y'$, which is the extension of $E \rightarrow Y$ with respect to the base change $Y' \rightarrow Y$, we see that

$$\# \left\{ y \in Y'(\mathbb{F}_{q^{dn}}) \mid 0 \leq \Theta(y) < \frac{\pi}{2} \right\} \geq \# \left\{ y \in Y(\mathbb{F}_{q^n}) \mid 0 \leq \Theta(y) < \frac{\pi}{2d} \right\} > \varepsilon q^n$$

where $\varepsilon > 0$ is some constant depending only on d . Now verbatim the same counting as above gives us a constant $c = c(E/Y, d)$, depending only on $E \rightarrow Y$ and d , such that $C' \rightarrow Y'$ has at least $c \cdot \frac{\log g}{\log \log g}$ singularities. \square

It follows that the genus g of such families of curves has to be bounded.

Theorem 5.6 (The genus is bounded). *Let $E \rightarrow Y$ be a non-isotrivial family of elliptic curves and $C' \rightarrow Y'$ a family of curves of genus g defined over a covering $Y' \rightarrow Y$ of degree at most d . Assume that the Jacobian $J' \rightarrow Y'$ of $C' \rightarrow Y'$ is Y' -isogenous to the g -fold product of $E' \rightarrow Y'$. Then the genus of $C' \rightarrow Y'$ is bounded, i. e. there is a constant $B = B(E/Y, d)$, depending only on $E \rightarrow Y$ and d , such that g is smaller than B .*

Proof. Without loss of generality, we may assume that $E \rightarrow Y$ has semistable reduction everywhere. If not, we can achieve this after a finite base change using the semistable reduction theorem.

Let $J' \rightarrow Y'$ be the family of Jacobians of $C' \rightarrow Y'$. By assumption $J' \rightarrow Y'$ is Y' -isogenous to the g -fold product of $E' \rightarrow Y'$. Hence, $J' \rightarrow Y'$ has semistable reduction and, therefore, the family of curves $C' \rightarrow Y'$ is semistable. So by (5.5) and (5.2) the number δ of singularities in the geometric fibers of $C \rightarrow Y$ satisfies (notice that $h(E') \leq d \cdot h(E)$)

$$c_0 \frac{\log g}{\log \log g} \cdot g \leq \delta \leq 12d \cdot h(E) \cdot g$$

where the constant $c_0 > 0$ depends only on $E \rightarrow Y$ and d . But then g cannot be arbitrarily large, since the left hand side is not linearly bounded by g . So there is a constant B depending only on $E \rightarrow Y$ and d such that g is smaller than B . \square

6. CONCLUSION

We come to the results announced in the introduction. Our base field is now \mathbb{C} again.

Theorem 6.1 (Bound for the genus). *Let $C \rightarrow Y$ be a family of curves of genus g whose Jacobian $J \rightarrow Y$ is Y -isogenous to the g -fold product of a non-isotrivial family of elliptic curves $E \rightarrow Y$ which can be defined over a number field. Then the genus g is bounded, i. e. there is a number $B = B(E/Y)$ depending only on $E \rightarrow Y$ such that g is smaller than B .*

Proof. Using Proposition (3.7) we see that after replacing Y by a finite covering, there is a covering $Y' \rightarrow Y$ of degree at most 2 such that $C' \rightarrow Y'$ can be defined over a number field F which depend only on $E \rightarrow Y$.

Theorem (4.7) tells us that there is a finite prime $\text{Spec } \mathbb{F}_q \rightarrow \text{Spec } \mathcal{O}_F$, whose choice depends only on $E \rightarrow Y$, such that the reduction of $C' \rightarrow Y'$ modulo this prime yields a family $\widetilde{C'} \rightarrow \widetilde{Y'}$ of curves of genus g over \mathbb{F}_q whose Jacobian is $\widetilde{Y'}$ -isogenous to the g -fold product of the reduction $\widetilde{E'} \rightarrow \widetilde{Y'}$ of $E \rightarrow Y$.

By Theorem (5.6) applied to $\widetilde{C'} \rightarrow \widetilde{Y'}$ there is a number B , depending only on $E \rightarrow Y$, such that g is smaller than B . \square

Coming to Shimura curves, we first give a finiteness statement about modular families of elliptic curves.

Proposition 6.2 (Finiteness of modular families of elliptic curves). *Fix two integers q and s . Then there are only finitely many semistable modular families of elliptic curves $E \rightarrow Y$ defined over a base curve Y of genus at most q and smooth outside a set $S \subset Y$ of cardinality at most s .*

Proof. Let $E \rightarrow Y$ be a modular family as in the proposition and let $Y \xrightarrow{j_E} \mathbb{P}_{\mathbb{C}}^1$ be the j -map corresponding to the family $E \rightarrow Y$. Because of the semistability of $E \rightarrow Y$, an application of the ABC-conjecture for function fields yields

$$\deg(j_E) \leq 6 \cdot (2q - 2 + s) =: d.$$

So, in particular, the degree of the j -map is absolutely bounded by d . Therefore, the modular family of elliptic curves $E \rightarrow Y$ is given by a subgroup $\Gamma \subset \text{SL}_2(\mathbb{Z})$ of index at most d .

Since $\mathrm{SL}_2(\mathbb{Z})$ is finitely generated, there are only finitely many subgroups Γ of $\mathrm{SL}_2(\mathbb{Z})$ of index at most d . Thus, we have only finitely many semistable modular families of elliptic curves $E \rightarrow Y$ over a curve of genus at most q and smooth outside a set of cardinality at most s . \square

Example 6.3. Let $q = 0$ so that $Y = \mathbb{P}_{\mathbb{C}}^1$. Beauville showed that for $s \leq 3$ there are no non-isotrivial semistable families of elliptic curves at all. And for $s = 4$, Beauville showed that there are exactly six non-isotrivial semistable families of elliptic curves, all modular, corresponding to the congruence subgroups $\Gamma(3)$, $\Gamma_1(4) \cap \Gamma(2)$, $\Gamma_1(5)$, $\Gamma_1(6)$, $\Gamma_0(8) \cap \Gamma_1(4)$ and $\Gamma_0(9) \cap \Gamma_1(3)$, see [Be82].

So, if the family of Jacobians is Y -isogenous to the g -fold product of a modular family of elliptic curves, then we get the following result from Theorem (6.1).

Corollary 6.4 (Uniform bound for modular families). *Fix two integers q and s . Then there is a constant $B = B(q, s)$ such that for any semistable family of curves $C \rightarrow Y$, which is defined over a base curve Y of genus at most q and whose family of Jacobians $J \rightarrow Y$ is smooth outside a set $S \subset Y$ of cardinality at most s and Y -isogenous to the g -fold product of a modular family of elliptic curves $E \rightarrow Y$, the genus g of the fibers of $C \rightarrow Y$ is bounded above by B . In particular, B depends only on q and s .*

Proof. By Proposition (6.2), there are only finitely many semistable modular families of elliptic curves $E \rightarrow Y$ over a curve of genus at most q and smooth outside a set of cardinality at most s . Because of the modularity, each one can be defined over some number field [De79].

So Theorem (6.1) gives for each $E \rightarrow Y$, a bound $B(E/Y)$ such that for g larger than $B(E/Y)$, the g -fold product of $E \rightarrow Y$ is not Y -isogenous to a Jacobian. Thus, taking $B = B(q, s)$ to be the maximum of these finitely many numbers $B(E/Y)$ proves the corollary. \square

We derive the following corollary.

Corollary 6.5 (Curves over $\mathbb{P}_{\mathbb{C}}^1$ with strict maximal Higgs field). *Given an integer $s \geq 0$, there is a natural number $B = B(s)$, depending only on s , such that for any semistable family of curves $C \rightarrow \mathbb{P}_{\mathbb{C}}^1$, whose Jacobian $J \rightarrow \mathbb{P}_{\mathbb{C}}^1$ is smooth outside a set $S \subset \mathbb{P}_{\mathbb{C}}^1$ of cardinality at most s and has a strictly maximal Higgs field, the genus g of the fibers of $C \rightarrow \mathbb{P}_{\mathbb{C}}^1$ is bounded above by the number B .*

Proof. Because of Corollary (2.3), we may choose $B = B(s)$ to be the constant $B(0, s)$ from Corollary (6.4). \square

In particular, a given rational Shimura curve parameterizing a family of high-dimensional Abelian varieties with strictly maximal Higgs field does not lie in the closure of the Schottky locus. This proves Theorem (1.1).

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